Unbounded subnormal weighted shifts on directed trees. II

Piotr Budzyński, Zenon Jan Jabłoński, Il Bong Jung, and Jan Stochel

Abstract. Criteria for subnormality of unbounded injective weighted shifts on leafless directed trees with one branching vertex are proposed. The case of classical weighted shifts is discussed. The relevance of an inductive limit approach to subnormality of weighted shifts on directed trees is revealed.

1. Introduction

This paper is concerned with unbounded subnormal weighted shifts on directed trees (see [**11**] for basic facts on weighted shifts on directed trees and [**5**] for the literature on subnormality). The only known general characterizations of subnormality of unbounded Hilbert space operators are due to Bishop and Foiaş [**4, 8**], and Szafraniec [**22**]. These characterizations refer to either semispectral measures or elementary spectral measures. They seem to be difficult to apply in the context of weighted shifts on directed trees, especially when we want them to be formulated in terms of weights. The situation is much better when the operators in question have invariant domains. For such operators the full characterization of subnormality has been given in [**19**] (see also [**6**] for a new simplifying approach). Using this abstract tool, we have invented in [**5**, Theorem 5.1.1] a new criterion (read: a sufficient condition) for subnormality of weighted shifts on directed trees written in terms of consistent systems of probability measures. The main objective of the present paper is to develop this idea in the context of directed trees $\mathscr{T}_{\eta,\kappa}$ which are models of leafless directed trees with one branching vertex (see Section 4). The characterizations of subnormality of bounded weighted shifts on $\mathscr{T}_{\eta,\kappa}$ given in [**11**, Corollary 6.2.2] (see also [**11**, Theorem 6.2.1]) have been deduced from the celebrated Lambert's theorem (cf. [**14**]) which is no longer valid for unbounded operators (even for weighted shifts on $\mathscr{T}_{\infty,\kappa}$, cf. [**12**]). However, as proved in Theorem 4.1, all but one criteria in [**11**, Corollary 6.2.2] remain valid in the unbounded case. The exception is essentially different (cf. Theorem 4.1). Under the assumption of

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determinacy, the criteria in Theorem 4.1 become full characterizations (see Theorem 4.3). An inductive limit approach to subnormality is shown to be applicable in the context of weighted shifts on directed trees (see Section 5).

This paper is a sequel to [**5**], to which we refer readers for more background, discussion and references.

2. Preliminaries

Let $\mathbb{Z}$ and $\mathbb{C}$ stand for the sets of integers and complex numbers respectively. Denote by $\mathbb{R}_+$ the set of all nonnegative real numbers. Set $\mathbb{Z}_+ = \{0,1,2,3,\ldots\}$ and $\mathbb{N} = \{1,2,3,4,\ldots\}$. We write $\mathfrak{B}(\mathbb{R}_+)$ for the σ-algebra of all Borel subsets of $\mathbb{R}_+$ and δ_0 for the Borel probability measure on $\mathbb{R}_+$ concentrated at 0. A sequence $\{t_n\}_{n=0}^\infty \subseteq \mathbb{R}_+$ is said to be a *Stieltjes moment sequence* if there exists a positive Borel measure μ on $\mathbb{R}_+$ such that $t_n = \int_0^\infty s^n \,\mathrm{d}\,\mu(s)$ for every $n \in \mathbb{Z}_+$, where $\int_0^\infty$ means the integral over $\mathbb{R}_+$; such μ is called a *representing measure* of $\{t_n\}_{n=0}^\infty$. We say that a Stieltjes moment sequence is *determinate* if it has only one representing measure. A two-sided sequence $\{t_n\}_{n=-\infty}^\infty \subseteq \mathbb{R}_+$ is said to be a *two-sided Stieltjes moment sequence* if there exists a positive Borel measure μ on $(0,\infty)$ such that $t_n = \int_{(0,\infty)} s^n \,\mathrm{d}\,\mu(s)$ for every $n \in \mathbb{Z}$; such μ is called a *representing measure* of $\{t_n\}_{n=-\infty}^\infty$. It follows from [**2**, page 202] (see also [**13**, Theorem 6.3]) that

(2.1) $\{t_n\}_{n=-\infty}^\infty \subseteq \mathbb{R}_+$ is a two-sided Stieltjes moment sequence if and only if $\{t_{n-k}\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $k \in \mathbb{Z}_+$.

We refer the reader to [**2, 18**] for the foundations of the theory of moment problems.

Let A be an operator in a complex Hilbert space $\mathcal{H}$ (all operators considered in this paper are linear). Denote by $\mathcal{D}(A)$ the domain of A. Set $\mathcal{D}^\infty(A) = \bigcap_{n=0}^\infty \mathcal{D}(A^n)$. A linear subspace $\mathcal{E}$ of $\mathcal{D}(A)$ is said to be a *core* of A if the graph of A is contained in the closure of the graph of the restriction $A|_{\mathcal{E}}$ of A to $\mathcal{E}$. A densely defined operator S in $\mathcal{H}$ is said to be *subnormal* if there exists a complex Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Sh = Nh$ for all $h \in \mathcal{D}(S)$. We refer the reader to [**3, 23**] for background on unbounded operators. We write $\mathrm{LIN}\,\mathcal{F}$ for the linear span of a subset $\mathcal{F}$ of $\mathcal{H}$.

Let $\mathscr{T} = (V,E)$ be a directed tree (V and E stand for the sets of vertices and edges of $\mathscr{T}$, respectively). Denote by root the root of $\mathscr{T}$ (provided it exists) and write $\mathsf{Root}(\mathscr{T}) = \{\mathsf{root}\}$ if $\mathscr{T}$ has a root and $\mathsf{Root}(\mathscr{T}) = \varnothing$ otherwise. Define $V^\circ = V \setminus \mathsf{Root}(\mathscr{T})$. Set $\mathsf{Chi}(u) = \{v \in V\colon (u,v) \in E\}$ for $u \in V$. A member of $\mathsf{Chi}(u)$ is called a *child* of u. Denote by par the partial function from V to V which assigns to each vertex $u \in V^\circ$ its parent $\mathsf{par}(u)$ (i.e. a unique $v \in V$ such that $(v,u) \in E$). Set $\mathsf{Des}(u) = \bigcup_{n=0}^\infty \{w \in V\colon \mathsf{par}^n(w) = u\}$ for $u \in V$. Note that the terms in the union are pairwise disjoint. We refer the reader to [**5, 11**] for all facts about directed trees needed in this paper.

Denote by $\ell^2(V)$ the Hilbert space of all square summable complex functions on V with the inner product $\langle f,g\rangle = \sum_{u\in V} f(u)\overline{g(u)}$. For $u \in V$, we define $e_u \in \ell^2(V)$ to be the characteristic function of the one-point set $\{u\}$. Then $\{e_u\}_{u\in V}$ is an orthonormal basis of $\ell^2(V)$. Set $\mathscr{E}_V = \mathrm{LIN}\{e_u\colon u \in V\}$. By a *weighted shift* on $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ} \subseteq \mathbb{C}$ we mean the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ defined by

$$\begin{aligned} \mathcal{D}(S_{\boldsymbol{\lambda}}) &= \{f \in \ell^2(V)\colon \varLambda_{\mathscr{T}} f \in \ell^2(V)\},\\ S_{\boldsymbol{\lambda}} f &= \varLambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\boldsymbol{\lambda}}), \end{aligned}$$

where $\varLambda_{\mathscr{T}}$ is the mapping defined on functions $f\colon V \to \mathbb{C}$ via

$$(\varLambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f\big(\mathsf{par}(v)\big) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}. \end{cases} \tag{2.2}$$

Suppose $\mathscr{E}_V \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$ and $u \in V$ is such that $\mathsf{Chi}(u) \neq \varnothing$ and $\{\|S_{\boldsymbol\lambda}^n e_v\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence with a representing measure μ_v for every $v \in \mathsf{Chi}(u)$. Then, following [**11, 5**], we say that $S_{\boldsymbol\lambda}$ satisfies the *consistency condition* at u if[1]

$$\sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{s} \,\mathrm{d}\,\mu_v(s) \leqslant 1, \tag{2.3}$$

Let us recall the main result of [**5**].

Theorem 2.1 ([**5**, Theorem 5.1.1]). *Assume that $\mathscr{E}_V \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$ and there exist a system $\{\mu_v\}_{v\in V}$ of Borel probability measures on $\mathbb{R}_+$ and a system $\{\varepsilon_v\}_{v \in V}$ of real numbers[2] that satisfy the following condition*

$$\mu_u(\sigma) = \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \int_\sigma \frac{1}{s} \,\mathrm{d}\,\mu_v(s) + \varepsilon_u \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ u \in V. \tag{2.4}$$

Then $S_{\boldsymbol\lambda}$ is subnormal.

Definition. We say that $u \in V$ is a *Stieltjes vertex* (with respect to $S_{\boldsymbol\lambda}$) if $e_u \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$ and $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence.

In many cases, if each child of u is a Stieltjes vertex, then so is u itself (cf. [**5**, Lemma 4.1.3]). If u is a Stieltjes vertex, then in general its children are not (cf. [**11**, Example 6.1.6]). However, assuming that u has only one child w and $\lambda_w \neq 0$, if u is a Stieltjes vertex, then so is w (see [**11**, Lemma 6.1.5] for the bounded case).

Lemma 2.2. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$ and let $u, w \in V$ be such that $\mathsf{Chi}(u) = \{w\}$. Suppose u is a Stieltjes vertex and $\lambda_w \neq 0$. Then w is a Stieltjes vertex and the following two assertions hold*:

(i) *the mapping $\mathscr{M}_w^{\mathrm{b}}(\boldsymbol\lambda) \ni \mu \to \rho_\mu \in \mathscr{M}_u(\boldsymbol\lambda)$ defined by*

$$\rho_\mu(\sigma) = |\lambda_w|^2 \int_\sigma \frac{1}{s} \,\mathrm{d}\,\mu(s) + \Big(1 - |\lambda_w|^2 \int_0^\infty \frac{1}{s} \,\mathrm{d}\,\mu(s)\Big)\delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

is a bijection with the inverse $\mathscr{M}_u(\boldsymbol\lambda) \ni \rho \to \mu_\rho \in \mathscr{M}_w^{\mathrm{b}}(\boldsymbol\lambda)$ given by

$$\mu_\rho(\sigma) = \frac{1}{|\lambda_w|^2} \int_\sigma s \,\mathrm{d}\,\rho(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

where $\mathscr{M}_w^{\mathrm{b}}(\boldsymbol\lambda)$ is the set of all representing measures μ of $\{\|S_{\boldsymbol\lambda}^n e_w\|^2\}_{n=0}^\infty$ such that $\int_0^\infty \frac{1}{s} \,\mathrm{d}\,\mu(s) \leqslant \frac{1}{|\lambda_w|^2}$, and $\mathscr{M}_u(\boldsymbol\lambda)$ is the set of all representing measures ρ of $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$,

(ii) *if the Stieltjes moment sequence $\{\|S_{\boldsymbol\lambda}^n e_w\|^2\}_{n=0}^\infty$ is determinate, then so are $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ and $\{\|S_{\boldsymbol\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$.*

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[1] We adhere to the standard convention that $0 \cdot \infty = 0$.

[2] Note that (2.4) implies that the system $\{\varepsilon_v\}_{v \in V}$ consists of nonnegative real numbers.

Proof. Since $e_u \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$, $\mathsf{Chi}(u) = \{w\}$ and $\lambda_w \neq 0$, we infer from [**11**, Proposition 3.1.3] that $e_w = \frac{1}{\lambda_w} S_{\boldsymbol\lambda} e_u \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$, and thus

$$\|S_{\boldsymbol\lambda}^n e_w\|^2 = \frac{1}{|\lambda_w|^2}\|S_{\boldsymbol\lambda}^{n+1} e_u\|^2, \quad n \in \mathbb{Z}_+.$$

This and [**5**, Lemma 2.4.1] with $\vartheta = 1$ and $t_n = \|S_{\boldsymbol\lambda}^{n+1} e_u\|^2$ complete the proof. □

3. Classical weighted shifts

In this section we focus on classical weighted shifts (see [**17, 15**] for both the bounded and the unbounded cases). As shown in [**11**, Remark 3.1.4], unilateral and bilateral weighted shifts can be regarded as weighted shifts on directed trees $(\mathbb{Z}_+, \{(n, n+1)\colon n \in \mathbb{Z}_+\})$ and $(\mathbb{Z}, \{(n, n+1)\colon n \in \mathbb{Z}\})$ respectively. The reader should be aware that we enumerate weights of unilateral and bilateral weighted shifts in accordance with our notation, i.e.,

$$S_{\boldsymbol\lambda} e_n = \lambda_{n+1} e_{n+1}, \quad n \in \mathbb{Z}_+ \ \text{(respectively: } n \in \mathbb{Z}). \tag{3.1}$$

Using our approach, we can derive the Berger-Gellar-Wallen criterion for subnormality of injective unilateral weighted shifts (see [**9, 10**] for the bounded case and [**19**, Theorem 4] for the unbounded one).

Theorem 3.1. *If $S_{\boldsymbol\lambda}$ is a unilateral weighted shift with nonzero weights $\boldsymbol\lambda = \{\lambda_n\}_{n=1}^\infty$ (cf. (3.1)), then the following three conditions are equivalent*:

(i) *$S_{\boldsymbol\lambda}$ is subnormal,*
(ii) *$(1, |\lambda_1|^2, |\lambda_1\lambda_2|^2, |\lambda_1\lambda_2\lambda_3|^2, \ldots)$ is a Stieltjes moment sequence,*
(iii) *k is a Stieltjes vertex for every $k \in \mathbb{Z}_+$.*

Proof. It is clear that $\mathscr{E}_V \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$.

(i)⇒(iii) Employ [**5**, Proposition 4.1.1].

(iii)⇒(ii) This is evident, because the sequence $(1, |\lambda_1|^2, |\lambda_1\lambda_2|^2, |\lambda_1\lambda_2\lambda_3|^2, \ldots)$ coincides with $\{\|S_{\boldsymbol\lambda}^n e_0\|^2\}_{n=0}^\infty$.

(ii)⇒(i) Let μ be a representing measure of the Stieltjes moment sequence $\{\|S_{\boldsymbol\lambda}^n e_0\|^2\}_{n=0}^\infty$ (which in general may not be determinate, cf. [**21**]). Define the sequence $\{\mu_n\}_{n=0}^\infty$ of Borel probability measures on $\mathbb{R}_+$ by

$$\mu_n(\sigma) = \frac{1}{\|S_{\boldsymbol\lambda}^n e_0\|^2}\int_\sigma s^n \,\mathrm{d}\,\mu(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, n \in \mathbb{Z}_+.$$

It is then clear that

$$\begin{aligned} \mu_0(\sigma) &= |\lambda_1|^2 \int_\sigma \frac{1}{s}\,\mathrm{d}\,\mu_1(s) + \mu(\{0\})\delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \\ \mu_n(\sigma) &= |\lambda_{n+1}|^2 \int_\sigma \frac{1}{s}\,\mathrm{d}\,\mu_{n+1}(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, n \in \mathbb{N}, \end{aligned}$$

which means that the systems $\{\mu_n\}_{n=0}^\infty$ and $\{\varepsilon_n\}_{n=0}^\infty := (\mu(\{0\}), 0, 0, \ldots)$ satisfy the assumptions of Theorem 2.1. This completes the proof. □

Now we prove an analogue of the Berger-Gellar-Wallen criterion for subnormality of injective bilateral weighted shifts (see [**7**, Theorem II.6.12] for the bounded case and [**19**, Theorem 5] for the unbounded one).

Theorem 3.2. *If $S_{\boldsymbol\lambda}$ is a bilateral weighted shift with nonzero weights $\boldsymbol\lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ (cf. (3.1)), then the following four conditions are equivalent*:

(i) *$S_{\boldsymbol\lambda}$ is subnormal,*
(ii) *the two-sided sequence $\{t_n\}_{n=-\infty}^{\infty}$ defined by*

$$t_n = \begin{cases} |\lambda_1 \cdots \lambda_n|^2 & \text{for } n \geqslant 1, \\ 1 & \text{for } n = 0, \\ |\lambda_{n+1} \cdots \lambda_0|^{-2} & \text{for } n \leqslant -1, \end{cases}$$

is a two-sided Stieltjes moment sequence,
(iii) *$-k$ is a Stieltjes vertex for infinitely many nonnegative integers k,*
(iv) *k is a Stieltjes vertex for every $k \in \mathbb{Z}$.*

Proof. Clearly, $\mathscr{E}_V \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$.
(i)⇒(iv) Employ [**5**, Proposition 4.1.1].
(iv)⇒(iii) Evident.
(iii)⇒(iv) Apply Lemma 2.2.
(iv)⇒(ii) Since $t_{n-k} = t_{-k}\|S_{\boldsymbol\lambda}^n e_{-k}\|^2$ for all $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$, we can apply the characterization (2.1).
(ii)⇒(i) Let μ be a representing measure of $\{t_n\}_{n=-\infty}^{\infty}$. Define the two-sided sequence $\{\mu_n\}_{n=-\infty}^{\infty}$ of Borel probability measures on $\mathbb{R}_+$ by (note that $\mu(\{0\}) = 0$)

$$\mu_n(\sigma) = \frac{1}{\|S_{\boldsymbol\lambda}^n e_0\|^2} \int_\sigma s^n \,\mathrm{d}\,\mu(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, n \in \mathbb{Z}.$$

We easily verify that

$$\mu_n(\sigma) = |\lambda_{n+1}|^2 \int_\sigma \frac{1}{s} \,\mathrm{d}\,\mu_{n+1}(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, n \in \mathbb{Z},$$

which means that the systems $\{\mu_n\}_{n=-\infty}^{\infty}$ and $\{\varepsilon_n\}_{n=-\infty}^{\infty}$ with $\varepsilon_n \equiv 0$ satisfy the assumptions of Theorem 2.1. This completes the proof. □

In view of Theorems 3.1 and 3.2, the necessary condition for subnormality of general operators given in [**5**, Proposition 3.2.1] becomes sufficient in the case of injective classical weighted shifts. To the best of our knowledge, the class of injective classical weighted shifts seems to be the only one[3] for which this phenomenon occurs regardless of whether or not the operators in question have sufficiently many qusi-analytic vectors (see [**20**] for more details; see also [**5**, Sections 3.2 and 5.3]).

4. One branching vertex

Our next aim is to discuss subnormality of weighted shifts with nonzero weights on directed trees that have only one *branching vertex*, i.e., a vertex with at least two children. By [**5**, Proposition 5.2.1], there is no loss of generality in assuming that the directed tree under consideration is countably infinite and leafless. Such directed trees are one step more complicated than those involved in the definitions of classical weighted shifts (see Section 3). Countably infinite and leafless directed trees with one branching vertex can be modelled as follows (see Figure 1). Given

[3] Not mentioning symmetric operators which are always subnormal, cf. [**1**, Appendix I.2].

$\eta, \kappa \in \mathbb{Z}_+ \sqcup \{\infty\}$ with $\eta \geqslant 2$, we define the directed tree $\mathscr{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ by[4]

$$\begin{aligned} V_{\eta,\kappa} &= \big\{ -k \colon k \in J_\kappa \big\} \sqcup \{0\} \sqcup \big\{(i,j) \colon i \in J_\eta,\, j \in \mathbb{N}\big\}, \\ E_{\eta,\kappa} &= E_\kappa \sqcup \big\{(0,(i,1)) \colon i \in J_\eta\big\} \sqcup \big\{((i,j),(i,j+1)) \colon i \in J_\eta,\, j \in \mathbb{N}\big\}, \\ E_\kappa &= \big\{(-k, -k+1) \colon k \in J_\kappa\big\}, \end{aligned}$$

where $J_\iota := \{k \in \mathbb{N} \colon k \leqslant \iota\}$ for $\iota \in \mathbb{Z}_+ \sqcup \{\infty\}$.

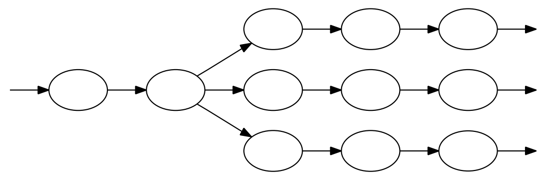

Figure 1

If $\kappa < \infty$, then the directed tree $\mathscr{T}_{\eta,\kappa}$ has the root $-\kappa$. If $\kappa = \infty$, then $\mathscr{T}_{\eta,\infty}$ is rootless. In all cases, 0 is the branching vertex of $\mathscr{T}_{\eta,\kappa}$.

Caution. One of the advantages of considering weighted shifts $S_{\boldsymbol\lambda}$ on $\mathscr{T}_{\eta,\kappa}$ is that $e_0 \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$ if and only if $\mathscr{E}_{V_{\eta,\kappa}} \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$.

We begin by showing that all but one criteria for subnormality of injective weighted shifts on $\mathscr{T}_{\eta,\kappa}$ given in [**11**, Corollary 6.2.2] remain valid in the unbounded case. The only exception is condition (iii) in Theorem 4.1 below which is an unbounded variant (essentially different) of condition (ii-b) in [**11**, Corollary 6.2.2]. Below, we adhere to the notation $\lambda_{i,j}$ instead of a more formal expression $\lambda_{(i,j)}$.

THEOREM 4.1. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ such that $e_0 \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$. Suppose that there exists a sequence $\{\mu_i\}_{i=1}^\eta$ of Borel probability measures on $\mathbb{R}_+$ such that*

$$\int_0^\infty s^n \,\mathrm{d}\,\mu_i(s) = \Big|\prod_{j=2}^{n+1} \lambda_{i,j}\Big|^2, \quad n \in \mathbb{N},\ i \in J_\eta. \tag{4.1}$$

Then $S_{\boldsymbol\lambda}$ is subnormal provided one of the following four conditions holds:

(i) $\kappa = 0$ *and*

$$\sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s} \,\mathrm{d}\,\mu_i(s) \leqslant 1, \tag{4.2}$$

(ii) $0 < \kappa < \infty$ *and*

$$\sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s} \,\mathrm{d}\,\mu_i(s) = 1, \tag{4.3}$$

$$\Big|\prod_{j=0}^{l-1} \lambda_{-j}\Big|^2 \sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s^{l+1}} \,\mathrm{d}\,\mu_i(s) = 1, \quad l \in J_{\kappa-1}, \tag{4.4}$$

$$\Big|\prod_{j=0}^{\kappa-1} \lambda_{-j}\Big|^2 \sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s^{\kappa+1}} \,\mathrm{d}\,\mu_i(s) \leqslant 1, \tag{4.5}$$

[4] The symbol "$\sqcup$" denotes disjoint union of sets.

(iii) $0<\kappa<\infty$ *and there exists a Borel probability measure* ν *on* $\mathbb{R}_+$ *such that*

$$\int_0^\infty s^n \,\mathrm{d}\,\nu(s) = \Big|\prod_{j=\kappa-n}^{\kappa-1} \lambda_{-j}\Big|^2, \quad n \in J_\kappa, \tag{4.6}$$

$$\int_\sigma s^\kappa \,\mathrm{d}\,\nu(s) = \Big|\prod_{j=0}^{\kappa-1} \lambda_{-j}\Big|^2 \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_\sigma \frac{1}{s}\,\mathrm{d}\,\mu_i(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{4.7}$$

(iv) $\kappa=\infty$ *and equalities* (4.3) *and* (4.4) *are satisfied.*

Proof. (i) Define the system of Borel probability measures $\{\mu_v\}_{v\in V_{\eta,0}}$ on $\mathbb{R}_+$ and the system $\{\varepsilon_v\}_{v\in V_{\eta,0}}$ of nonnegative real numbers by

$$\mu_0(\sigma) = \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_\sigma \frac{1}{s}\,\mathrm{d}\,\mu_i(s) + \varepsilon_0\delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

$$\varepsilon_0 = 1 - \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s}\,\mathrm{d}\,\mu_i(s),$$

and

$$\begin{cases} \mu_{i,n}(\sigma) = \dfrac{1}{\|S_{\boldsymbol\lambda}^{n-1} e_{i,1}\|^2} \displaystyle\int_\sigma s^{n-1}\,\mathrm{d}\,\mu_i(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ i\in J_\eta,\ n\in\mathbb{N}, \\ \varepsilon_{i,n} = 0, \quad i \in J_\eta,\ n \in \mathbb{N}. \end{cases} \tag{4.8}$$

(We write $\mu_{i,j}$ and $\varepsilon_{i,j}$ instead of the more formal expressions $\mu_{(i,j)}$ and $\varepsilon_{(i,j)}$.) Clearly $\mu_{i,1}=\mu_i$ for all $i\in J_\eta$. Using (4.1) and (4.2), we verify that the systems $\{\mu_v\}_{v\in V_{\eta,0}}$ and $\{\varepsilon_v\}_{v\in V_{\eta,0}}$ are well-defined and satisfy the assumptions of Theorem 2.1. Hence $S_{\boldsymbol\lambda}$ is subnormal.

(ii) Define the systems $\{\mu_v\}_{v\in V_{\eta,\kappa}}$ and $\{\varepsilon_v\}_{v\in V_{\eta,\kappa}}$ by (4.8) and

$$\mu_0(\sigma) = \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_\sigma \frac{1}{s}\,\mathrm{d}\,\mu_i(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{4.9}$$

$$\mu_{-l}(\sigma) = \Big|\prod_{j=0}^{l-1} \lambda_{-j}\Big|^2 \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_\sigma \frac{1}{s^{l+1}}\,\mathrm{d}\,\mu_i(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ l\in J_{\kappa-1}, \tag{4.10}$$

$$\mu_{-\kappa}(\sigma) = \Big|\prod_{j=0}^{\kappa-1} \lambda_{-j}\Big|^2 \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_\sigma \frac{1}{s^{\kappa+1}}\,\mathrm{d}\,\mu_i(s) + \varepsilon_{-\kappa}\delta_0(\sigma),\ \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{4.11}$$

$$\varepsilon_v = \begin{cases} 0 & \text{if } v \in V_{\eta,\kappa}^\circ, \\ 1 - \Big|\prod_{j=0}^{\kappa-1}\lambda_{-j}\Big|^2 \sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s^{\kappa+1}}\,\mathrm{d}\,\mu_i(s) & \text{if } v=-\kappa. \end{cases} \tag{4.12}$$

Applying (4.1), (4.3), (4.4) and (4.5), we verify that the systems $\{\mu_v\}_{v\in V_{\eta,\kappa}}$ and $\{\varepsilon_v\}_{v\in V_{\eta,\kappa}}$ are well-defined and satisfy the assumptions of Theorem 2.1. Therefore $S_{\boldsymbol\lambda}$ is subnormal.

(iii) First note that $\|S_{\boldsymbol\lambda}^n e_{-\kappa}\|^2 = \Big|\prod_{j=\kappa-n}^{\kappa-1}\lambda_{-j}\Big|^2$ for $n\in J_\kappa$. Define the systems $\{\mu_v\}_{v\in V_{\eta,\kappa}}$ and $\{\varepsilon_v\}_{v\in V_{\eta,\kappa}}$ by (4.8) and

$$\mu_{-l}(\sigma) = \frac{1}{\|S_{\boldsymbol\lambda}^{-l+\kappa} e_{-\kappa}\|^2}\int_\sigma s^{-l+\kappa}\,\mathrm{d}\,\nu(s), \quad \sigma\in\mathfrak{B}(\mathbb{R}_+),\ l\in J_\kappa\cup\{0\},$$

$$\varepsilon_v = \begin{cases} 0 & \text{if } v \in V_{\eta,\kappa}^{\circ}, \\ \nu(\{0\}) & \text{if } v = -\kappa. \end{cases}$$

Clearly $\mu_{-\kappa} = \nu$, which together with (4.1), (4.6) and (4.7) implies that the systems $\{\mu_v\}_{v\in V_{\eta,\kappa}}$ and $\{\varepsilon_v\}_{v\in V_{\eta,\kappa}}$ satisfy the assumptions of Theorem 2.1. As a consequence, $S_{\boldsymbol\lambda}$ is subnormal.

(iv) Define the system $\{\mu_v\}_{v\in V_{\eta,\kappa}}$ by (4.8), (4.9) and (4.10). Arguing as in the proof of (ii), we see that the systems $\{\mu_v\}_{v\in V_{\eta,\kappa}}$ and $\{\varepsilon_v\}_{v\in V_{\eta,\kappa}}$ with $\varepsilon_v \equiv 0$ satisfy the assumptions of Theorem 2.1, and so $S_{\boldsymbol\lambda}$ is subnormal. □

It turns out that conditions (ii) and (iii) of Theorem 4.1 are equivalent without assuming that (4.1) is satisfied.

LEMMA 4.2. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^{\circ}}$ such that $e_0 \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$ and let $\{\mu_i\}_{i=1}^{\eta}$ be a sequence of Borel probability measures on $\mathbb{R}_+$. Then conditions* (ii) *and* (iii) *of Theorem* 4.1 (*with the same* κ) *are equivalent.*

PROOF. (ii)⇒(iii) Let $\{\mu_{-l}\}_{l=0}^{\kappa}$ be the Borel probability measures on $\mathbb{R}_+$ defined by (4.9), (4.10) and (4.11) with $\varepsilon_{-\kappa}$ given by (4.12). Set $\nu = \mu_{-\kappa}$. It follows from (4.11) that for every $n \in J_\kappa$,

$$\int_\sigma s^n \,\mathrm{d}\,\nu(s) = \Big|\prod_{j=0}^{\kappa-1} \lambda_{-j}\Big|^2 \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_\sigma \frac{1}{s^{\kappa+1-n}} \,\mathrm{d}\,\mu_i(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{4.13}$$

This immediately implies (4.7). By (4.9), (4.10) and (4.13), we have

$$\int_\sigma s^n \,\mathrm{d}\,\nu(s) = \begin{cases} \dfrac{|\prod_{j=0}^{\kappa-1} \lambda_{-j}|^2}{|\prod_{j=0}^{\kappa-n-1} \lambda_{-j}|^2} \mu_{-(\kappa-n)}(\sigma) & \text{if } n \in J_{\kappa-1}, \\ |\prod_{j=0}^{\kappa-1} \lambda_{-j}|^2 \,\mu_0(\sigma) & \text{if } n = \kappa, \end{cases}$$

for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. Substituting $\sigma = \mathbb{R}_+$ and using the fact that $\{\mu_{-l}\}_{l=0}^{\kappa-1}$ are probability measures, we obtain (4.6).

(iii)⇒(ii) Given $n \in J_\kappa$, we define the positive Borel measure ρ_n on $\mathbb{R}_+$ by $\rho_n(\sigma) = \int_\sigma s^n \,\mathrm{d}\,\nu(s)$ for $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. By (4.7), equality (4.13) holds for $n = \kappa$. If this equality holds for a fixed $n \in J_\kappa \setminus \{1\}$, then $\rho_n(\{0\}) = 0$ and consequently

$$\int_\sigma s^{n-1} \,\mathrm{d}\,\nu(s) = \int_\sigma \frac{1}{s} \,\mathrm{d}\,\rho_n(s) \overset{(4.13)}{=} \Big|\prod_{j=0}^{\kappa-1} \lambda_{-j}\Big|^2 \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_\sigma \frac{1}{s^{\kappa+1-(n-1)}} \,\mathrm{d}\,\mu_i(s)$$

for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. Hence, by reverse induction on n, (4.13) holds for all $n \in J_\kappa$. Substituting $\sigma = \mathbb{R}_+$ into (4.13) and using (4.6), we obtain (4.3) and (4.4). It follows from (4.13), applied to $n = 1$, that for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,

$$\begin{aligned} \nu(\sigma) &= \nu(\sigma \setminus \{0\}) + \nu(\{0\})\delta_0(\sigma) = \int_\sigma \frac{1}{s} \,\mathrm{d}\,\rho_1(s) + \nu(\{0\})\delta_0(\sigma) \\ &\overset{(4.13)}{=} \Big|\prod_{j=0}^{\kappa-1} \lambda_{-j}\Big|^2 \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_\sigma \frac{1}{s^{\kappa+1}} \,\mathrm{d}\,\mu_i(s) + \nu(\{0\})\delta_0(\sigma). \end{aligned} \tag{4.14}$$

Substituting $\sigma = \mathbb{R}_+$ into (4.14) and using the fact that $\nu(\mathbb{R}_+) = 1$, we obtain (4.5). This completes the proof. □

Under the assumption of determinacy, the sufficient conditions for subnormality appearing in Theorem 4.1 become necessary (see also Remark 4.4 below).

THEOREM 4.3. *Let $S_{\boldsymbol\lambda}$ be a subnormal weighted shift on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^{\circ}}$. If $e_0 \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$ and*

$$\Big\{\sum_{i=1}^{\eta}\Big|\prod_{j=1}^{n+1}\lambda_{i,j}\Big|^2\Big\}_{n=0}^{\infty} \text{ is a determinate Stieltjes moment sequence,} \tag{4.15}$$

then the following four assertions hold:

(i) *if $\kappa = 0$, then there exists a sequence $\{\mu_i\}_{i=1}^{\eta}$ of Borel probability measures on $\mathbb{R}_+$ that satisfy* (4.1) *and* (4.2),

(ii) *if $0 < \kappa < \infty$, then there exists a sequence $\{\mu_i\}_{i=1}^{\eta}$ of Borel probability measures on $\mathbb{R}_+$ that satisfy* (4.1), (4.3), (4.4) *and* (4.5),

(iii) *if $0 < \kappa < \infty$, then there exist a sequence $\{\mu_i\}_{i=1}^{\eta}$ of Borel probability measures on $\mathbb{R}_+$ and a Borel probability measure ν on $\mathbb{R}_+$ that satisfy* (4.1), (4.6) *and* (4.7),

(iv) *if $\kappa = \infty$, then there exists a sequence $\{\mu_i\}_{i=1}^{\eta}$ of Borel probability measures on $\mathbb{R}_+$ that satisfy* (4.1), (4.3) *and* (4.4).

Moreover, if [5] $\sum_{n=1}^{\infty}\big(\sum_{i=1}^{\eta}\big|\prod_{j=1}^{n}\lambda_{i,j}\big|^2\big)^{-1/2n} = \infty$, *then* (4.15) *is satisfied.*

PROOF. It is easily seen that

$$\|S_{\boldsymbol\lambda}^{n+1}e_0\|^2 = \sum_{i=1}^{\eta}\Big|\prod_{j=1}^{n+1}\lambda_{i,j}\Big|^2, \quad n \in \mathbb{Z}_+. \tag{4.16}$$

By [**5**, Proposition 4.1.1], for every $u \in V_{\eta,\kappa}$ the sequence $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence. For each $i \in J_\eta$, we choose a representing measure μ_i of $\{\|S_{\boldsymbol\lambda}^n e_{i,1}\|^2\}_{n=0}^\infty$. It is easily seen that (4.1) holds. Since, by (4.15) and (4.16), the Stieltjes moment sequence $\{\|S_{\boldsymbol\lambda}^{n+1}e_0\|^2\}_{n=0}^\infty$ is determinate, we infer from [**5**, Lemma 4.1.3], applied to $u = 0$, that (4.2) holds and $\{\|S_{\boldsymbol\lambda}^n e_0\|^2\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence with the representing measure μ_0 given by

$$\mu_0(\sigma) = \sum_{i=1}^{\eta}|\lambda_{i,1}|^2\int_\sigma \frac{1}{s}\,\mathrm{d}\,\mu_i(s) + \varepsilon_0\delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{4.17}$$

where ε_0 is a nonnegative real number. In view of the above, assertion (i) is proved.

Suppose $0 < \kappa \leqslant \infty$. Since $\{\|S_{\boldsymbol\lambda}^n e_0\|^2\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence, we deduce from Lemma 2.2, applied to $u = -1$, that $\{\|S_{\boldsymbol\lambda}^{n+1}e_{-1}\|^2\}_{n=0}^\infty$ and $\{\|S_{\boldsymbol\lambda}^n e_{-1}\|^2\}_{n=0}^\infty$ are determinate Stieltjes moment sequences and

$$\int_0^\infty \frac{1}{s}\,\mathrm{d}\,\mu_0(s) \leqslant \frac{1}{|\lambda_0|^2}, \tag{4.18}$$

$$\mu_{-1}(\sigma) = |\lambda_0|^2\int_\sigma \frac{1}{s}\,\mathrm{d}\,\mu_0(s) + \varepsilon_{-1}\delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{4.19}$$

where μ_{-1} is the representing measure of $\{\|S_{\boldsymbol\lambda}^n e_{-1}\|^2\}_{n=0}^\infty$ and ε_{-1} is a nonnegative real number. Inequality (4.18) combined with equality (4.17) implies that $\varepsilon_0 = 0$ and therefore that (4.5) holds for $\kappa = 1$. Substituting $\sigma = \mathbb{R}_+$ into (4.17), we obtain

[5] Equivalently, by (4.16), e_0 is a quasi-analytic vector of $S_{\boldsymbol\lambda}$ (see [**16**] for the definition).

(4.3). This completes the proof of assertion (ii) for $\kappa = 1$. Note also that equalities (4.17) and (4.19), combined with $\varepsilon_0 = 0$, yield

$$\mu_{-1}(\sigma) = |\lambda_0|^2 \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_\sigma \frac{1}{s^2} \,\mathrm{d}\,\mu_i(s) + \varepsilon_{-1}\delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

If $\kappa > 1$, then arguing by induction, we conclude that for every $k \in J_\kappa$ the Stieltjes moment sequences $\{\|S_{\boldsymbol\lambda}^{n+1} e_{-k}\|^2\}_{n=0}^\infty$ and $\{\|S_{\boldsymbol\lambda}^{n} e_{-k}\|^2\}_{n=0}^\infty$ are determinate and

$$\mu_{-l}(\sigma) = \Big|\prod_{j=0}^{l-1} \lambda_{-j}\Big|^2 \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_\sigma \frac{1}{s^{l+1}} \,\mathrm{d}\,\mu_i(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, l \in J_{\kappa-1}, \tag{4.20}$$

where μ_{-l} is the representing measure of $\{\|S_{\boldsymbol\lambda}^{n} e_{-l}\|^2\}_{n=0}^\infty$. Substituting $\sigma = \mathbb{R}_+$ into (4.20), we obtain (4.4). This completes the proof of assertion (iv). Finally, if $1 < \kappa < \infty$, then again by Lemma 2.2, now applied to $u = -\kappa$, we have $\int_0^\infty \frac{1}{s} \,\mathrm{d}\,\mu_{-\kappa+1}(s) \leqslant \frac{1}{|\lambda_{-\kappa+1}|^2}$. This inequality together with (4.20) yields (4.5), which completes the proof of assertion (ii).

(iii) can be deduced from (ii) via Lemma 4.2.

To show the "moreover" part, we can argue as in the proof of [**5**, Theorem 5.3.1] (see also Footnote 5). This completes the proof. □

Remark 4.4. A careful look at the proof reveals that Theorem 4.3 remains valid if instead of assuming that $S_{\boldsymbol\lambda}$ is subnormal, we assume that u is a Stieltjes vertex for every $u \in \big\{-k\colon k \in J_\kappa\big\} \sqcup \{0\} \sqcup \mathsf{Chi}(0)$.

Corollary 4.5. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ such that $e_0 \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$. Suppose that v is a Stieltjes vertex for every $v \in \big\{-k\colon k \in J_\kappa\big\} \sqcup \{0\} \sqcup \mathsf{Chi}(0)$ and that $\{\|S_{\boldsymbol\lambda}^{n+1} e_0\|^2\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence. Then the following three assertions hold*:

(i) *$S_{\boldsymbol\lambda}$ is subnormal,*
(ii) *$\{\|S_{\boldsymbol\lambda}^{n+1} e_{-j}\|^2\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence for every integer j such that $0 \leqslant j \leqslant \kappa$,*
(iii) *$S_{\boldsymbol\lambda}$ satisfies the consistency condition* (2.3) *at the vertex $u = -j$ for every integer j such that $0 \leqslant j \leqslant \kappa$.*

Proof. (i) In view of Remark 4.4, there exists a sequence $\{\mu_i\}_{i=1}^\eta$ of Borel probability measures on $\mathbb{R}_+$ satisfying (4.1) and one of the conditions (i), (ii) and (iv) of Theorem 4.1. Hence, by Theorem 4.1, $S_{\boldsymbol\lambda}$ is subnormal.

(ii) See the proof of Theorem 4.3.

(iii) Apply (ii) and [**5**, Lemma 4.1.3 (ii)]. □

Remark 4.6. Note that the consistency condition (2.3) at a vertex u depends on the choice of the system $\{\mu_v\}_{v \in \mathsf{Chi}(u)}$ of representing measures μ_v of Stieltjes moment sequences $\{\|S_{\boldsymbol\lambda}^{n} e_v\|^2\}_{n=0}^\infty$. However, by [**5**, Lemma 4.1.3], if the Stieltjes moment sequence $\{\|S_{\boldsymbol\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$ is determinate, then the consistency condition (2.3) at u is independent of the choice of $\{\mu_v\}_{v \in \mathsf{Chi}(u)}$, i.e., it is satisfied for every system $\{\mu_v\}_{v \in \mathsf{Chi}(u)}$ of representing measures μ_v of Stieltjes moment sequences $\{\|S_{\boldsymbol\lambda}^{n} e_v\|^2\}_{n=0}^\infty$. This and assertion (ii) of Corollary 4.5 justify not mentioning explicitly representing measures in assertion (iii) of this corollary.

5. Subnormality via subtrees

Proposition 5.2 below shows that the study of subnormality of weighted shifts on rootless directed trees can be reduced, in a sense, to the case of directed trees with root. Theorem 2.1, which is our main criterion for subnormality of weighted shifts on directed trees, seems to be inapplicable in this context. However, we can employ an inductive limit approach.

THEOREM 5.1. *Let S be a densely defined operator in a complex Hilbert space $\mathcal{H}$. Suppose that there are a family $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ of closed linear subspaces of $\mathcal{H}$ and an upward directed family $\{\mathcal{X}_\omega\}_{\omega\in\Omega}$ of subsets of $\mathcal{H}$ such that*

(i) *$\mathcal{X}_\omega \subseteq \mathcal{D}^\infty(S)$ and $S^n(\mathcal{X}_\omega) \subseteq \mathcal{H}_\omega$ for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$,*
(ii) *$\mathcal{F}_\omega := \mathrm{LIN}\bigcup_{n=0}^\infty S^n(\mathcal{X}_\omega)$ is dense in $\mathcal{H}_\omega$ for every $\omega \in \Omega$,*
(iii) *$S|_{\mathcal{F}_\omega}$ is a subnormal operator in $\mathcal{H}_\omega$ for every $\omega \in \Omega$,*
(iv) *$\mathcal{F} := \mathrm{LIN}\bigcup_{n=0}^\infty S^n\big(\bigcup_{\omega\in\Omega}\mathcal{X}_\omega\big)$ is a core of S.*

Then S is subnormal.

PROOF. The families $\{\mathcal{F}_\omega\}_{\omega\in\Omega}$ and $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ are upward directed, $S(\mathcal{F}_\omega) \subseteq \mathcal{F}_\omega$ for all $\omega \in \Omega$, $\mathcal{F} = \bigcup_{\omega\in\Omega}\mathcal{F}_\omega$ and $S(\mathcal{F}) \subseteq \mathcal{F}$. Hence, we can argue as in the proof of [**5**, Theorem 3.1.1] by using [**6**, Theorem 21]. □

Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Note that if $u \in V$, then the space $\ell^2(\mathsf{Des}(u))$ (which is regarded as a closed linear subspace of $\ell^2(V)$) is invariant for $S_{\boldsymbol\lambda}$, i.e.,

$$S_{\boldsymbol\lambda}\big(\mathcal{D}(S_{\boldsymbol\lambda}) \cap \ell^2(\mathsf{Des}(u))\big) \subseteq \ell^2(\mathsf{Des}(u)). \tag{5.1}$$

(For this, apply (2.2) and the inclusion $\mathsf{par}\big(V \setminus \big(\mathsf{Des}(u) \cup \mathsf{Root}(\mathscr{T})\big)\big) \subseteq V \setminus \mathsf{Des}(u)$.) Denote by $S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(u))}$ the operator in $\ell^2(\mathsf{Des}(u))$ given by $\mathcal{D}(S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(u))}) = \mathcal{D}(S_{\boldsymbol\lambda}) \cap \ell^2(\mathsf{Des}(u))$ and $S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(u))}f = S_{\boldsymbol\lambda}f$ for $f \in \mathcal{D}(S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(u))})$. It is easily seen that $S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(u))}$ coincides with the weighted shift on the rooted directed tree $(\mathsf{Des}(u), (\mathsf{Des}(u) \times \mathsf{Des}(u)) \cap E)$ with weights $\{\lambda_v\}_{v\in\mathsf{Des}(u)\setminus\{u\}}$ (see [**11**, Proposition 2.1.8] for more details on this and related subtrees).

We are now ready to apply the aforementioned inductive limit procedure to prove the unbounded counterpart of [**11**, Corollary 6.1.4].

PROPOSITION 5.2. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on a rootless directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Suppose that $\mathscr{E}_V \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$. If Ω is a subset of V such that $V = \bigcup_{\omega\in\Omega}\mathsf{Des}(\omega)$, then the following conditions are equivalent*:

(i) *$S_{\boldsymbol\lambda}$ is subnormal,*
(ii) *for every $\omega \in \Omega$, $S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(\omega))}$ is subnormal as an operator acting in $\ell^2(\mathsf{Des}(\omega))$.*

PROOF. (ii)⇒(i) Using an induction argument and (5.1) one can show that $S_{\boldsymbol\lambda}^n e_v \in \ell^2(\mathsf{Des}(v)) \subseteq \ell^2(\mathsf{Des}(u))$ for all $n \in \mathbb{Z}_+$, $v \in \mathsf{Des}(u)$ and $u \in V$. Hence

$$\mathcal{X}_\omega := \mathrm{LIN}\big\{e_v\colon v \in \mathsf{Des}(\omega)\big\} \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda}) \text{ and } S_{\boldsymbol\lambda}^n(\mathcal{X}_\omega) \subseteq \ell^2(\mathsf{Des}(\omega))$$

for all $\omega \in \Omega$ and $n \in \mathbb{Z}_+$. It follows from [**11**, Proposition 2.1.4] and the equality $V = \bigcup_{\omega\in\Omega}\mathsf{Des}(\omega)$ that for each pair $(\omega_1, \omega_2) \in \Omega \times \Omega$, there exists $\omega \in \Omega$ such that $\mathsf{Des}(\omega_1) \cup \mathsf{Des}(\omega_2) \subseteq \mathsf{Des}(\omega)$, and thus $\{\mathcal{X}_\omega\}_{\omega\in\Omega}$ is an upward directed family of subsets of $\ell^2(V)$. By applying [**11**, Proposition 3.1.3] and Theorem 5.1 to $S = S_{\boldsymbol\lambda}$ and $\mathcal{H}_\omega = \ell^2(\mathsf{Des}(\omega))$, we deduce that $S_{\boldsymbol\lambda}$ is subnormal.

The reverse implication (i)⇒(ii) is obvious because $\mathcal{X}_\omega \subseteq \mathcal{D}(S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(\omega))})$. □

It follows from [**11**, Proposition 2.1.6] that if $\mathscr{T}$ is a rootless directed tree, then $V = \bigcup_{k=1}^{\infty} \mathsf{Des}(\mathsf{par}^k(u))$ for every $u \in V$, and so the set Ω in Proposition 5.2 may always be chosen to be countably infinite.

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Katedra Zastosowań Matematyki, Uniwersytet Rolniczy w Krakowie, ul. Balicka 253c, PL-30198 Kraków

E-mail address: piotr.budzynski@ur.krakow.pl

Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków, Poland

E-mail address: Zenon.Jablonski@im.uj.edu.pl

Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea

E-mail address: ibjung@knu.ac.kr

Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków, Poland

E-mail address: Jan.Stochel@im.uj.edu.pl